

SOME COMPLETELY MONOTONIC PROPERTIES FOR THE (p, q)-GAMMA FUNCTION

VALMIR KRASNIQI AND FATON MEROVCI

ABSTRACT. It is defined $\Gamma_{p,q}$ function, a generalize of Γ function. Also, we defined $\psi_{p,q}$ -analogue of the psi function as the log derivative of $\Gamma_{p,q}$. For the $\Gamma_{p,q}$ -function, are given some properties related to convexity, log-convexity and completely monotonic function. Also, some properties of $\psi_{p,q}$ analog of the ψ function have been established. As an application, when $p \rightarrow \infty, q \rightarrow 1$, we obtain all result of [12] and [21].

1. INTRODUCTION

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [3]):

$$(1) \quad \psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)},$$

where $\gamma = 0.57721 \dots$ denotes Euler's constant.

Euler, gave another equivalent definition for the $\Gamma(x)$ (see [2], [19])

$$(2) \quad \Gamma_p(x) = \frac{p! p^x}{x(x+1) \cdot \dots \cdot (x+p)} = \frac{p^x}{x(1 + \frac{x}{1}) \cdot \dots \cdot (1 + \frac{x}{p})}, \quad x > 0,$$

where p is positive integer, and

$$(3) \quad \Gamma(x) = \lim_{p \rightarrow \infty} \Gamma_p(x).$$

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The p -analogue of the psi function as the logarithmic derivative of the Γ_p function (see [12]), that is

$$(4) \quad \psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma'_p(x)}{\Gamma_p(x)}.$$

The following representations are valid:

$$(5) \quad \Gamma_p(x) = \int_0^p \left(1 - \frac{t}{p}\right)^p t^{x-1} dt,$$

$$(6) \quad \psi_p(x) = \ln p - \int_0^\infty \frac{e^{-xt}(1 - e^{-(p+1)t})}{1 - e^{-t}} dt$$

and

$$(7) \quad \psi_p^{(m)}(x) = (-1)^{m+1} \int_0^\infty \frac{t^m \cdot e^{-xt}}{1 - e^{-t}} (1 - e^{-(p+1)t}) dt.$$

Jackson (see [8–10, 20]) defined the q -analogue of the gamma function as

$$(8) \quad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1,$$

and

$$(9) \quad \Gamma_q(x) = \frac{(q^{-1}; q^{-1})_\infty}{(q^{-x}; q^{-1})_\infty} (q - 1)^{1-x} q^{\binom{x}{2}}, \quad q > 1,$$

where $(a; q)_\infty = \prod_{j=0}^\infty (1 - aq^j)$.

The q -gamma function has the following integral representation (see [4])

$$\Gamma_q(t) = \int_0^\infty x^{t-1} E_q^{-qx} d_q x,$$

where $E_q^x = \sum_{j=0}^\infty q^{\frac{j(j-1)}{2}} \frac{x^j}{[j]!} = (1 + (1 - q)x)_q^\infty$, which is the q -analogue of the classical exponential function. The q -analogue of the psi function is defined for $0 < q < 1$ as the logarithmic derivative of the q -gamma function, that is, $\psi_q(x) = \frac{d}{dx} \log \Gamma_q(x)$. Many properties of the q -gamma function were derived by Askey [4]. It is well known that $\Gamma_q(x) \rightarrow \Gamma(x)$ and $\psi_q(x) \rightarrow \psi(x)$ as $q \rightarrow 1^-$. From (8), for $0 < q < 1$ and $x > 0$ we get

$$(10) \quad \psi_q(x) = -\log(1 - q) + \log q \sum_{n \geq 0} \frac{q^{n+x}}{1 - q^{n+x}} = -\log(1 - q) + \log q \sum_{n \geq 1} \frac{q^{nx}}{1 - q^n},$$

and from (9) for $q > 1$ and $x > 0$ we obtain

$$(11) \quad \begin{aligned} \psi_q(x) &= -\log(q - 1) + \log q \left(x - \frac{1}{2} - \sum_{n \geq 0} \frac{q^{-n-x}}{1 - q^{-n-x}} \right) \\ &= -\log(q - 1) + \log q \left(x - \frac{1}{2} - \sum_{n \geq 1} \frac{q^{-nx}}{1 - q^{-n}} \right). \end{aligned}$$

A Stieltjes integral representation for $\psi_q(x)$ with $0 < q < 1$ is given in [16]. It is well-known that ψ' is strictly completely monotonic on $(0, \infty)$, that is,

$$(-1)^n(\psi'(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0,$$

see [3, Page 260]. From (10) and (11) we conclude that ψ'_q has the same property for any $q > 0$,

$$(-1)^n(\psi'_q(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0.$$

If $q \in (0, 1)$, using the second representation of $\psi_q(x)$ given in (10) can be shown that

$$(12) \quad \psi_q^{(k)}(x) = \log^{k+1} q \sum_{n \geq 1} \frac{n^k \cdot q^{nx}}{1 - q^n},$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 1$, for all $k \geq 1$. If $q > 1$, from the second representation of $\psi_q(x)$ given in (11) we obtain

$$(13) \quad \psi'_q(x) = \log q \left(1 + \sum_{n \geq 1} \frac{nq^{-nx}}{1 - q^{-nx}} \right),$$

and for $k \geq 2$,

$$(14) \quad \psi_q^{(k)}(x) = (-1)^{k-1} \log^{k+1} q \sum_{n \geq 1} \frac{n^k q^{-nx}}{1 - q^{-nx}}.$$

Hence, $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 0$, for all $q > 1$.

Definition 1.1. For $x > 0$, $p \in \mathbb{N}$ and for $q \in (0, 1)$

$$(15) \quad \Gamma_{p,q}(x) = \frac{[p]_q^x [p]_q!}{[x]_q [x+1]_q \cdots [x+p]_q},$$

where $[p]_q = \frac{1-q^p}{1-q}$.

It is easy to see than $\Gamma_{p,q}(x)$ fit into the following commutative diagrams:

$$\begin{array}{ccc} \Gamma_{p,q}(x) & \xrightarrow{p \rightarrow \infty} & \Gamma_q(x) \\ \downarrow q \rightarrow 1 & & \downarrow q \rightarrow 1 \\ \Gamma_p(x) & \xrightarrow{p \rightarrow \infty} & \Gamma(x) \end{array}$$

We define (p, q) -analogue of the psi function as the logarithmic derivative of the p, q -gamma function, that is,

$$(16) \quad \psi_{p,q}(x) = \frac{d}{dx} \log \Gamma_{p,q}(x).$$

Definition 1.2. *The function f is called log-convex if for all $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and for all $x, y > 0$ the following inequality holds*

$$\log f(\alpha x + \beta y) \leq \alpha \log f(x) + \beta \log f(y),$$

or equivalently

$$f(\alpha x + \beta y) \leq (f(x))^\alpha \cdot (f(y))^\beta.$$

Now, we will give some definitions about completely monotonic function:

A function f is said to be *completely monotonic* on an open interval I , if f has derivatives of all orders on I and satisfies

$$(17) \quad (-1)^n f^{(n)}(x) \geq 0, (x \in I, n = 0, 1, 2, \dots).$$

If the inequality (17) is strict, then f is said to be *strictly completely monotonic* on I .

A positive function f is said to be *logarithmically completely monotonic* (see [18]) on an open interval I , if f satisfies

$$(18) \quad (-1)^n [\ln f(x)]^{(n)} \geq 0, (x \in I, n = 1, 2, \dots).$$

If the inequality (18) is strict, then f is said to be *strictly logarithmically completely monotonic*.

Let C and L denote the set of completely monotonic functions and the set of logarithmically completely monotonic functions, respectively. The relationship between completely monotonic functions and logarithmically completely monotonic functions can be presented (see [2]) by $L \subset C$.

The following theorem gives an integral characterization of completely monotone functions.

Theorem 1.1. (*Hausdorff- Bernstein- Widder Theorem*) *A function $\varphi : [0, \infty) \rightarrow R$ is completely monotone on $[0, \infty)$ if and only if it is the Laplace transform of a finite non-negative Borel measure μ on $[0, \infty)$, i.e., φ is of the form*

$$(19) \quad \varphi(r) = \int_0^\infty e^{-rt} d\mu(t).$$

- A non-negative finite linear combination of completely monotone functions is completely monotone.
- The product of two completely monotone functions is completely monotone.

2. MAIN RESULTS

Lemma 2.1. *For $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ we have:*

$$(20) \quad [1 + x]_q^\alpha [1 + y]_q^\beta \leq [1 + \alpha x + \beta y]_q.$$

Proof. From Youngs inequality

$$(21) \quad [x]_q^\alpha [y]_q^\beta \leq \alpha [x]_q + \beta [y]_q,$$

we have:

$$\begin{aligned} [1+x]_q^\alpha [1+y]_q^\beta &\leq \alpha [1+x]_q + \beta [1+y]_q \\ &= \alpha \left(\frac{1-q^{1+x}}{1-q} \right) + \beta \left(\frac{1-q^{1+y}}{1-q} \right) \\ &= \frac{1}{1-q} \left[1 - (\alpha q^{1+x} + \beta q^{1+y}) \right] \end{aligned}$$

We have to prove:

$$(22) \quad \alpha q^{1+x} + \beta q^{1+y} \geq q^{1+\alpha x + \beta y}.$$

From Youngs inequality we have:

$$q^{1+\alpha x + \beta y} = q((q^x)^\alpha (q^y)^\beta) \leq q(\alpha q^x + \beta q^y) = \alpha q^{1+x} + \beta q^{1+y}.$$

□

Theorem 2.2. *The function*

$$\Gamma_{p,q}(x) = \frac{[p]_q^x [p]_q!}{[x]_q [x+1]_q \cdots [x+p]_q}$$

is log-convex.

Proof. We have to prove that for all $\alpha, \beta > 0, \alpha + \beta = 1, x, y > 0$

$$(23) \quad \log \Gamma_{p,q}(\alpha x + \beta y) \leq \alpha \log \Gamma_{p,q}(x) + \beta \log \Gamma_{p,q}(y),$$

which is equivalent to

$$(24) \quad \Gamma_{p,q}(\alpha x + \beta y) \leq (\Gamma_{p,q}(x))^\alpha \cdot (\Gamma_{p,q}(y))^\beta.$$

By lemma 2.1 we obtain:

$$(25) \quad \left[1 + \frac{x}{k} \right]_q^\alpha \cdot \left[1 + \frac{y}{k} \right]_q^\beta \leq \alpha \left[1 + \frac{x}{k} \right]_q + \beta \left[1 + \frac{y}{k} \right]_q = \left[1 + \frac{\alpha x + \beta y}{k} \right]_q$$

for all $k \geq 1, k \in \mathbf{N}$.

Multiplying (25) for $k = 1, 2, \dots, p$ one obtains

$$\left[1 + \frac{x}{1} \right]_q^\alpha \cdots \left[1 + \frac{x}{p} \right]_q^\alpha \cdot \left[1 + \frac{y}{1} \right]_q^\beta \cdots \left[1 + \frac{y}{p} \right]_q^\beta \leq \left[1 + \frac{\alpha x + \beta y}{1} \right]_q \cdots \left[1 + \frac{\alpha x + \beta y}{p} \right]_q.$$

Now, taking the reciprocal values and multiplying by $[p]_q^{\alpha x + \beta y}$ one obtains (24) and thus the proof is completed. □

Lemma 2.3. *a) The function $\psi_{p,q}$ defined by (16) has the following series representation and integral representation*

$$(26) \quad \psi_{p,q}(x) = \ln[p]_q + \log q \sum_{k=0}^p \frac{q^{x+k}}{1 - q^{x+k}},$$

$$(27) \quad \psi_{p,q}(x) = \ln[p]_q - \int_0^\infty \frac{e^{-xt}}{1 - e^{-t}} (1 - e^{-pt}) d\gamma_q(t).$$

where $\gamma_q(t)$ is a discrete measure with positive masses $-\log q$ at the positive points $-k \log q, k = 1, 2, \dots, i.e.$

$$(28) \quad \gamma_q(t) = -\log q \sum_{k=1}^\infty \delta(t + k \log q), \quad 0 < q < 1.$$

b) The function $\psi_{p,q}$ is increasing on $(0, \infty)$.

c) The function $\psi'_{p,q}$ is strictly completely monotonic on $(0, \infty)$.

Proof. a) After logarithmical and derivative of (15) we take (26).

Using $\int_0^\infty e^{-xt} d\gamma_q(t) = \frac{-q^x \log q}{1 - q^x}, 0 < q < 1$ (see [7]) we take (27).

b) Let $0 < x < y$. Using (26) we obtain

$$\begin{aligned} \psi_{p,q}(x) - \psi_{p,q}(y) &= \log q \sum_{k=0}^p \left(\frac{1 - q^{y+k}}{q^{y+k}} - \frac{1 - q^{x+k}}{q^{x+k}} \right) \\ &= \log q \sum_{k=0}^p \left(\frac{q^x - q^{x+y+k} - q^y + q^{x+y+k}}{q^{x+y+k}} \right) \\ &= \log q \sum_{k=0}^p \left(\frac{q^x - q^y}{q^{x+y+k}} \right) < 0. \end{aligned}$$

c) Deriving n times the relation (26) one finds that:

$$(29) \quad \psi_{p,q}^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} (1 - e^{-(p+1)t}) d\gamma_q(t).$$

Hence $(-1)^n (\psi'_{p,q}(x))^{(n)} > 0$, for $x > 0, n \geq 0$. □

Remark 2.4. $\psi_{p,q}(x)$ fit into the following commutative diagrams

$$\begin{array}{ccc} \psi_{p,q}(x) & \xrightarrow{p \rightarrow \infty} & \psi_q(x) \\ \downarrow q \rightarrow 1 & & \downarrow q \rightarrow 1 \\ \psi_p(x) & \xrightarrow{p \rightarrow \infty} & \psi(x) \end{array}$$

3. LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION

Theorem 3.1. *The function $G_{p,q}(x; a_1, b_1, \dots, a_n, b_n)$ given by*

$$(30) \quad G_{p,q}(x) = G_{p,q}(x; a_1, b_1, \dots, a_n, b_n) = \prod_{i=1}^n \frac{\Gamma_{p,q}(x + a_i)}{\Gamma_{p,q}(x + b_i)}, \quad q \in (0, 1)$$

is a completely monotonic function on $(0, \infty)$, for any a_i and b_i , $i = 1, 2, \dots, n$, real numbers such that $0 < a_1 \leq \dots \leq a_n$, $0 < b_1 \leq b_2 \leq \dots \leq b_n$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1, 2, \dots, n$.

Proof. Let $h(x) = \sum_{i=1}^n (\log \Gamma_{p,q}(x + b_i) - \log \Gamma_{p,q}(x + a_i))$. Then for $k \geq 0$ we have

$$\begin{aligned} (-1)^k (h'(x))^{(k)} &= (-1)^k \sum_{i=1}^n (\psi_{p,q}^{(k)}(x + b_i) - \psi_{p,q}^{(k)}(x + a_i)) \\ &= (-1)^k \sum_{i=1}^n (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} (1 - e^{-(p+1)t}) (e^{-bi} - e^{-ai}) d\gamma_q(t) \\ &= (-1)^{2k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} (1 - e^{-(p+1)t}) \sum_{i=1}^n (e^{-bi} - e^{-ai}) d\gamma_q(t). \end{aligned}$$

Alzer [1] showed that if f is a decreasing and convex function on R , then holds

$$(31) \quad \sum_{i=1}^n f(b_i) \leq \sum_{i=1}^n f(a_i).$$

Thus, since the function $z \mapsto e^{-z}$, $z > 0$ is decreasing and convex on R , we have that $\sum_{i=1}^n (e^{-ai} - e^{-bi}) \geq 0$, so $(-1)^k (G'_{p,q}(x))^{(k)} \geq 0$ for $k \geq 0$. Hence h' is completely monotonic on $(0, \infty)$. Using the fact that if h' is completely monotonic function on $(0, \infty)$, then $\exp(-h)$ is also completely monotonic function on $(0, \infty)$ (see [5]), we get the desired result. \square

Theorem 3.2. *The function*

$$(32) \quad f(x) = \frac{1}{\left(\frac{[p]_q}{[p+1]_q} \Gamma_{p,q}(x) \right)^{\frac{1}{x}}}$$

is logarithmically completely monotonic in $(0, \infty)$.

Proof. Using Leibnitz rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(k)}(x) v^{(n-k)}(x),$$

we obtain

$$\begin{aligned}
[\ln f(x)]^{(n)} &= \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{x}\right)^{(k)} \left(-\ln \Gamma_{p,q}(x+1)\right)^{(n-k)} \\
&= -\frac{1}{x^{n+1}} \sum_{k=0}^n \binom{n}{k} (-1)^k k! x^{n-k} \psi_{p,q}^{(n-k-1)}(x+1) \\
&= -\frac{1}{x^{n+1}} g(x)
\end{aligned}$$

and

$$\begin{aligned}
g'(x) &= \sum_{k=0}^n \binom{n}{k} (-1)^k k! (n-k) x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1) + \\
&\quad + \sum_{k=0}^n \binom{n}{k} (-1)^k k! x^{n-k} \psi_{p,q}^{(n-k)}(x+1) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k! (n-k) x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1) + \\
&\quad + x^n \psi_{p,q}^{(n)}(x+1) + \sum_{k=0}^n \binom{n}{k} (-1)^k k! x^{n-k} \psi_{p,q}^{(n-k)}(x+1) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k! (n-k) x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1) + \\
&\quad + x^n \psi_{p,q}^{(n)}(x+1) + \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^{k+1} (k+1)! x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1) \\
&= \sum_{k=0}^{n-1} \left[\binom{n}{k} (n-k) - \binom{n}{k+1} (k+1) \right] (-1)^k k! x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1) \\
&\quad + x^n \psi_{p,q}^{(n)}(x+1) = x^n \psi_{p,q}^{(n)}(x+1).
\end{aligned}$$

If n is odd, then for $x > 0$,

$$\begin{aligned}
g'(x) > 0 &\Rightarrow g(x) > g(0) = 0 \Rightarrow (\ln f(x))^{(n)} < 0 \Rightarrow \\
&\Rightarrow (-1)^n (\ln f(x))^{(n)} > 0.
\end{aligned}$$

If n is even, then for $x > 0$,

$$\begin{aligned}
g'(x) < 0 &\Rightarrow g(x) < g(0) = 0 \Rightarrow (\ln f(x))^{(n)} > 0 \Rightarrow \\
&\Rightarrow (-1)^n (\ln f(x))^{(n)} > 0.
\end{aligned}$$

Hence,

$$(-1)^n (\ln f(x))^{(n)} > 0$$

for all real $x \in (0, \infty)$ and all integers $n \geq 1$. The proof is completed. \square

Remark 3.3. Let p tend to ∞ , then we obtain Theorem 1 of [6]. Let q tend to 1, then we obtain Theorem 2.1 of [14].

Let s and t be two real numbers with $s \neq t$, $\alpha = \min\{s, t\}$ and $\beta \geq -\alpha$, for $x \in (-\alpha, \alpha)$, define

$$h_{\beta,p,q}(x) = \begin{cases} \left[\frac{\Gamma_{p,q}(\beta+t)}{\Gamma_{p,q}(\beta+s)} \cdot \frac{\Gamma_{p,q}(x+s)}{\Gamma_{p,q}(x+t)} \right]^{\frac{1}{x-\beta}} & x \neq \beta \\ \exp[\psi_{p,q}(\beta+s) - \psi_{p,q}(\beta+t)] & x = \beta \end{cases}$$

The following theorem is a generalization of a result of [15].

Theorem 3.4. The function $h_{\beta,p,q}(x)$ is logarithmically completely monotonic on $(-\alpha, +\infty)$ if $s > t$.

Proof. For $x \neq \beta$, taking logarithm of the function $h_{\beta,p,q}(x)$ gives

$$\begin{aligned} \ln h_{\beta,p,q}(x) &= \frac{1}{x-\beta} \left[\ln \frac{\Gamma_{p,q}(\beta+t)}{\Gamma_{p,q}(\beta+s)} - \ln \frac{\Gamma_{p,q}(x+s)}{\Gamma_{p,q}(x+t)} \right] \\ &= \frac{\ln \Gamma_{p,q}(x+s) - \ln \Gamma_{p,q}(\beta+s)}{x-\beta} - \frac{\ln \Gamma_{p,q}(x+t) - \ln \Gamma_{p,q}(\beta+t)}{x-\beta} \\ &= \frac{1}{x-\beta} \int_{\beta}^x \psi_{p,q}(u+s) du - \frac{1}{x-\beta} \int_{\beta}^x \psi_{p,q}(u+t) du \\ &= \frac{1}{x-\beta} \int_{\beta}^x [\psi_{p,q}(u+s) - \psi_{p,q}(u+t)] du \\ &= \frac{1}{x-\beta} \int_{\beta}^x \int_t^s \psi'_{p,q}(u+v) dv du \\ &= \frac{1}{x-\beta} \int_{\beta}^x \varphi_{p,q,s,t}(u) du \\ &= \int_0^1 \varphi_{p,q,s,t}((x-\beta)u + \beta) du, \end{aligned}$$

and by differentiating $\ln h_{\beta,p,q}(x)$ with respect to x ,

$$(33) \quad [\ln h_{\beta,p,q}(x)]^{(k)} = \int_0^1 u^k \varphi_{p,q,s,t}^{(k)}((x-\beta)u + \beta) du.$$

If $x = \beta$ formula (33) is valid. Since functions $\psi'_{p,q}$ and $\varphi_{p,q,s,t}$ are completely monotonic in $(0, \infty)$ and $(-t, \infty)$ respectively, then $(-1)^i [\varphi_{p,q,s,t}(x)]^{(i)} \geq 0$ holds for $n \in (-t, \infty)$ for any nonnegative integer i . Thus

$$(-1)^k [\ln h_{\beta,p,q}(x)]^{(k)} = \int_0^1 u^k (-1)^k \varphi_{p,q,s,t}^{(k)}((x - \beta)u + \beta) du \geq 0$$

in $(-t, \infty)$ for $k \in N$. The proof is completed. \square

4. APPLICATION OF $\Gamma_{p,q}(x)$ FUNCTION

In the following, we give the $\Gamma_{p,q}$ analogue of results from [21]. Since the proofs are almost similar, we omit them.

Lemma 4.1. *Let a, b, c, d, e be real numbers such that $a + bx > 0$, $d + ex > 0$ and $a + bx \leq d + ex$. Then*

$$(34) \quad \psi_{p,q}(a + bx) - \psi_{p,q}(d + ex) \leq 0.$$

Lemma 4.2. *Let a, b, c, d, e, f be real numbers such that $a + bx > 0, d + ex > 0, a + bx \leq d + ex$ and $ef \geq bc > 0$. If*

(i) $\psi_{p,q}(a + bx) > 0$ or

(ii) $\psi_{p,q}(d + ex) > 0$

then

$$(35) \quad bc\psi_{p,q}(a + bx) - ef\psi_{p,q}(d + ex) \leq 0.$$

Lemma 4.3. *Let a, b, c, d, e, f be real numbers such that $a + bx > 0, d + ex > 0, a + bx \leq d + ex$ and $bc \geq ef > 0$. If*

(i) $\psi_{p,q}(d + ex) < 0$ or

(ii) $\psi_{p,q}(a + bx) < 0$

then

$$(36) \quad bc\psi_{p,q}(a + bx) - ef\psi_{p,q}(d + ex) \leq 0.$$

Theorem 4.4. *Let f_1 be a function defined by*

$$(37) \quad f_1(x) = \frac{\Gamma_{p,q}(a + bx)^c}{\Gamma_{p,q}(d + ex)^f}, \quad x \geq 0$$

where a, b, c, d, e, f are real numbers such that: $a + bx > 0, d + ex > 0, a + bx \leq d + ex, ef \geq bc > 0$. If $\psi_{p,q}(a + bx) > 0$ or $\psi_{p,q}(d + ex) > 0$ then the function f_1 is decreasing for $x \geq 0$ and for $x \in [0, 1]$ the following double inequality holds:

$$(38) \quad \frac{\Gamma_{p,q}(a + b)^c}{\Gamma_{p,q}(d + e)^f} \leq \frac{\Gamma_{p,q}(a + bx)^c}{\Gamma_{p,q}(d + ex)^f} \leq \frac{\Gamma_{p,q}(a)^c}{\Gamma_{p,q}(d)^f}.$$

In a similar way, using Lemma 4.3, it is easy to prove the following Theorem.

Theorem 4.5. Let f_1 be a function defined by

$$(39) \quad f_1(x) = \frac{\Gamma_{p,q}(a+bx)^c}{\Gamma_{p,q}(d+ex)^f}, \quad x \geq 0,$$

where a, b, c, d, e, f are real numbers such that: $a+bx > 0, d+ex > 0, a+bx \leq d+ex, bc \geq ef > 0$. If $\psi_{p,q}(d+ex) < 0$ or $\psi_{p,q}(a+bx) < 0$ then the function f_1 is decreasing for $x \geq 0$ and for $x \in [0, 1]$ the inequality (38) holds.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES, UNIVERSITY OF PRISHTINA, REPUBLIC OF KOSOVA

E-mail address: vali.99@hotmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PRISHTINA, REPUBLIC OF KOSOVA

E-mail address: fmerovci@yahoo.com